

Stationary quantum statistics of a non-Markovian atom laser

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We present a steady state analysis of a quantum-mechanical model of an atom laser. A single-mode atomic trap coupled to a continuum of external modes is driven by a saturable pumping mechanism. In the dilute flux regime, where atom-atom interactions are negligible in the output, we have been able to solve this model without making the Born-Markov approximation. The more exact treatment has a different effective damping rate and occupation of the lasing mode, as well as a shifted frequency and linewidth of the output. We examine gravitational damping numerically, finding linewidths and frequency shifts for a range of pumping rates. We treat mean field damping analytically, finding a memory function for the Thomas-Fermi regime. The occupation and linewidth are found to have a nonlinear scaling behavior which has implications for the stability of atom lasers.

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INTRODUCTION

The development of Bose-Einstein condensation (BEC) [1] has opened up the study of the atom laser: a coherent source of atoms analogous to the optical laser [2]. These atomic sources may demonstrate many of the advantageous properties of optical lasers, such as spatial and temporal coherence, and high spectral flux. This paper examines a quantum mechanical model of an atom laser consisting of a single-mode BEC coupled to a continuous output field, which may be a valid model for stable lasing systems when they have reached steady state. We find a compact form for the standard quantum limit for such a laser model which can be determined without making the Born-Markov approximation. This leads to nonlinear scaling of the steady-state occupation of the lasing mode with pumping rate, which has implications for the design of linewidth narrowing quantum feedback schemes and for the stability of atom lasers.

Outcoupling from a BEC has been achieved using radio frequency (rf) radiation [3, 4] and Raman transitions [5] to change the internal state of the atoms to a non-trapped state. Coupling the atoms out more slowly reduces the linewidth of the output at the expense of reducing the beam flux [6]. Optical lasers achieve a narrow linewidth through a competition between a saturable pumping mechanism and the damping of the lasing mode. This allows a higher pumping rate to increase both the total flux and the spectral density of the output. An atom laser with gain narrowing must also have a saturable pumping mechanism that operates at the same time as the damping [7]. A recent experiment managed to combine a continuously produced series of condensates to maintain a BEC in a trap indefinitely [8]. This procedure was an excellent first step towards providing a continuous pumping mechanism. The final step will be to produce a stimulated transition into the final BEC, which will allow mode-locking and reduce

the phase diffusion of the lasing mode. There have been several proposals for producing this kind of transition [9], but they have not yet been experimentally achieved.

The analogy between optical lasers and atom lasers has some limits. Theoretical examination of the properties of the output of atom lasers is complicated by the slow dispersion of atoms and the high interactions in atomic systems, which means that in general they exhibit multimode behaviour and the Born-Markov approximation cannot be made [10, 11, 12]. Atom laser models have made either a single-mode approximation for the lasing state or the mean-field approximation [1]. The first class of models cannot examine the multimode behaviour of the laser, which has been shown to have significant effects on the stability of the device [12]. The second class cannot examine the linewidth of an atom laser, which is a function of the quantum statistics of the lasing mode [13]. Many have also made the Born-Markov approximation, which has been shown to be invalid. This paper examines the output properties of an atom laser which has reached steady state, which means that the quantum statistics of the lasing mode must be included in the model. In order to avoid a multimode quantum field analysis, we assume that the steady state BEC can be modelled by a single mode, although we make no assumptions in our formalism about the spatial structure of that mode.

THE MODEL

The atom laser is modelled by separating it into three parts. Recent semiclassical modelling has shown that multimode behaviour of the BEC can be very important, often determining the stability of the system [12]. Once conditions for stability have been achieved, by definition we expect the system to converge to a stationary state. While the details of this sta-

tionary state will depend on the trapping conditions, pumping method and damping rate, we will assume in this work that in the steady state the BEC can be adequately described by a single mode. This single-mode BEC, also called the 'lasing' or 'system' mode, is modelled by the annihilation(creation) operator $a^{(\dagger)}$ and Hamiltonian H_s . Since the external field and trapped atoms are in a different electronic state, the free atoms are not necessarily affected by the trapping potential. We describe the continuum of external modes with field operators $c_p^{(\dagger)}$ and Hamiltonian H_o . The label p is a parameterization of the output eigenstates which in general may be degenerate. The coupling between the lasing mode and the output modes will be described by the interaction Hamiltonian H_i . At this stage, we will describe the pump which irreversibly couples the atoms from the pump reservoir into the trap mode by H_p . The total Hamiltonian is then written

$$H_{tot} = H_p + H_s + H_i + H_o \quad (1)$$

where

$$H_s = \hbar\omega_o a^\dagger a, \quad (2)$$

$$H_i = i\hbar\sqrt{\gamma} \int dp [\kappa^*(p, t) a^\dagger c_p - \kappa(p, t) a c_p^\dagger], \quad (3)$$

$$H_o = \int dp \hbar\omega_p c_p^\dagger c_p, \quad (4)$$

and the operators satisfy

$$[a, a^\dagger] = 1, \quad (5)$$

$$[c_p, c_{p'}^\dagger] = \delta(p - p') \quad (6)$$

with all other commutators zero. The natural description of the output field in the dilute regime is in the basis of energy eigenstates of the output Hamiltonian. The output field eigenfunctions satisfy

$$H_o|u_p\rangle = \hbar\omega_p|u_p\rangle. \quad (7)$$

In the position representation the effective coupling function between the trapped mode and the output field is the product of the condensate wave function and the spatial profile of the coupling field. We denote the effective outcoupled atomic state ket by $|\kappa, t\rangle = \int dp \kappa(p, t)|u_p\rangle$. We have taken out the overall coupling rate $\sqrt{\gamma}$ in the interaction Hamiltonian so that $\kappa(p, t)$ satisfies $\int dp |\kappa(p, t)|^2 = 1$. The system is described in Fig. 1.

To carry out a stationary analysis we assume the coupling amplitude is constant in time, apart from the Rabi frequency of the coupling which may be eliminated by moving into a rotating frame. This corresponds to a constant system wavefunction and spatial profile of the coupling.

Reversible output coupling

Atomic output coupling has been achieved through Raman transitions and radio frequency coupling of the trapped atoms

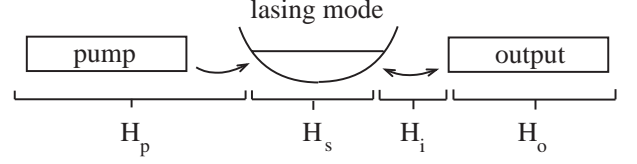


FIG. 1: Schematic of the atom laser model.

to an un-trapped or anti-trapped magnetic sublevel [3, 4, 5]. Later, we will consider the rf-coupling since in practice the energy shift due to rf-coupling is negligible, unlike the Raman process which imparts a significant impulse during coupling. Such an impulse is modelled in our formalism by shifting the center of coupling function κ in momentum space. Any realistic model of an atom laser should describe the coupling process in full as the usual Born and Markov approximations which are so useful in quantum optics are not generally applicable to atomic output couplers [10]. In making the Markov approximation for the atomic output coupler, one effectively assumes that the output process is irreversible, so that outcoupled atoms have only a minor influence on the system dynamics. While this works well in the optical case, the inertia of atoms alters the coupling process in a crucial way. An atom may be coupled into an untrapped internal state, but because it will linger in the interaction region for some time, and it may evolve for a short time in the output potential, and then undergo a transition back into the trapped state. The slow dissipation of atoms may cause the occupation of the system mode to increase.

We wish to solve the equations of motion of this model in the stationary long time limit. To clarify matters we will first reiterate some elementary results. The second-quantized output field operator for the output field can be expanded over the orthogonal set of modes

$$\hat{\psi}(x) = \int dp \langle x|u_p\rangle c_p. \quad (8)$$

Transforming into an interaction picture leaving only H_i simplifies the description of the output field. The interaction picture operators are

$$\hat{\psi}_I(x, t) = e^{iH_o(t-t_o)/\hbar} \hat{\psi}(x, t_o) e^{-iH_o(t-t_o)/\hbar}, \quad (9)$$

$$a_I(t) = e^{i(H_s+H_p)(t-t_o)/\hbar} a(t_o) e^{-i(H_s+H_p)(t-t_o)/\hbar}. \quad (10)$$

We set $t_o = 0$ in the rest of this work. The Green's function propagator for the output field is

$$G(x, y; t) = [\hat{\psi}_I(x, t), \hat{\psi}_I^\dagger(y, 0)] = \langle x|e^{-iH_o t/\hbar}|y\rangle, \quad (11)$$

where

$$\hat{\psi}_I(x, t) = \int dp \langle x|u_p\rangle c_p(0) e^{-i\omega_p t}. \quad (12)$$

Equations of motion without pumping

The memory function for the output beam is defined by [17]

$$F(x, t) \equiv \langle x | e^{-iH_o t/\hbar} | \kappa \rangle = \int dy G(x, y; t) \kappa(y). \quad (13)$$

This may be interpreted as the output wavefunction, after evolving for time t in the output field. The Heisenberg equations of motion for the output and trap operators are

$$\frac{dc_p^\dagger(t)}{dt} = i\omega_p c_p^\dagger(t) - \sqrt{\gamma} \kappa(p)^* a^\dagger(t) \quad (14)$$

$$\begin{aligned} \frac{da^\dagger(t)}{dt} = & -\frac{i}{\hbar} [a^\dagger(t), H_p] + i\omega_o a^\dagger(t) \\ & + \sqrt{\gamma} \int dp \kappa(p) c_p^\dagger(t). \end{aligned} \quad (15)$$

Using the formal solution of (14) with the Heisenberg picture version of (8), leads to the linear integral equation for the output field

$$\hat{\psi}(x, t) = \hat{\psi}_I(x, t) - \sqrt{\gamma} \int_0^t dt' a(t') F(x, t - t'). \quad (16)$$

This shows how the output field may be constructed if the evolution of the trap operator is known. This simple form for the output field arises because we are considering the case of a dilute output beam.

The output coupling is reversible so we require a description of the way atoms are coupled back into the trap. The overlap of the output wavefunction with the trap mode describes the probability of this occurring, and is defined by [11]

$$f(t) \equiv \langle \kappa | e^{-iH_o t/\hbar} | \kappa \rangle \quad (17)$$

$$= \int dx \int dy G(x, y; t) \kappa^*(x) \kappa(y). \quad (18)$$

The Greens function for the output potential of H_o and the form of the coupling completely describes the influence of the output on the system evolution. In the limit where $f(t)$ becomes a delta function, and the resulting equations of motion will be effectively memoryless or Markovian. This corresponds to the quantum optical limit of the coupling where photon loss from an optical cavity may be well described by a very narrow memory function [14].

Continuous pumping

The biggest hurdle in the development of a mode-locked, gain-narrowed atom laser is the production of a continuous pumping mechanism which can repopulate the lasing mode by a stimulated transition. The best experiments at this stage

have managed to merge independently produced condensates in a pulsed, but quasi-continuous manner, which provides an excellent starting point but does not involve the stimulated transition [8]. The stimulated nature of the transition is necessary for two reasons. The phase of the lasing mode must be preserved, and the small linewidth of a laser is a product of the competition between a damping process and a saturable, Bose-enhanced pumping [7]. It is a highly non-trivial experimental problem to design and build such a pumping process, and rather than address these issues, we will use a model process that should exhibit the qualitatively similar features to any successful continuous pumping process. A pumping process which can satisfy these requirements can be found in a model where cooled atoms in an excited state are passed over the trap containing the lasing mode [6]. The photon emission of the atoms would be stimulated by the presence of the highly occupied ground state, and they will make a transition into that state and emit a photon [15]. For a sufficiently optically thin sample, which can be made possible by having a very tight, effectively low dimensional trap, the photon is unlikely to be reabsorbed, and the process is effectively irreversible.

If the pump reservoir is sufficiently isolated from the cavity and external fields, and the pumping process is designed to be irreversible, then we may trace over the pump reservoir states to produce a master equation term for the reduced density matrix which describes only the cavity and external fields.

We choose to model an optical cooling process rather than the more experimentally successful evaporative cooling process as we are particularly interested in designing a continuously pumped system with a steady state. The reader is referred to the literature for the details of such proposals [6, 11, 15]. After we have traced over the pump modes, a term due to the effect of the pump in the equations of motion for $\langle a^\dagger(t) \rangle \langle a^\dagger a \rangle(t)$ is obtained by tracing over the trap modes in the number state basis [11]. It is important to note that we are not tracing over the output field modes, so the reduced density matrix spans the output field as well as the trap field. The term due to the pump in the master equation for the density matrix is

$$(\dot{\rho})_{pump} = r \mathcal{D}[a^\dagger] (n_s + \mathcal{A}[a^\dagger])^{-1} \rho \quad (19)$$

where r is the saturated gain rate and n_s is the saturation boson number. The superoperators are

$$\mathcal{A}[c] = \frac{1}{2} (c^\dagger c \rho - \rho c^\dagger c),$$

$$\mathcal{D}[c] \rho = c \rho c^\dagger - \mathcal{A} \rho.$$

This is the same pumping term as derived by Wiseman [16]. In the trap number state basis we denote $\rho_{n,m} = \langle n | \rho | m \rangle$, and the pump term becomes

$$\begin{aligned} (\dot{\rho}_{n,m})_{pump} = & r \frac{\sqrt{nm}}{n_s + (n+m)/2} \rho_{n-1, m-1} \\ & - r \frac{(n+m+2)/2}{n_s + (n+m+2)/2} \rho_{n, m}. \end{aligned} \quad (20)$$

This leads to the term in the $\langle a^\dagger a \rangle$ equation

$$\begin{aligned} \frac{d}{dt} \langle a^\dagger a \rangle \Big|_{\text{pump}} &= r \sum_{n=0}^{\infty} \frac{n+1}{n+1+n_s} \rho_{n,n} \\ &= r \sum_{n=0}^{\infty} \left(\frac{n}{n+n_s} + \frac{n_s}{(n+n_s)(n+1+n_s)} \right) \rho_{n,n} \\ &= r \left(\frac{\bar{N}}{\bar{N}+n_s} + O\left(\frac{n_s}{\bar{N}^2}\right) \right) \end{aligned} \quad (21)$$

The last line is obtained by assuming that when the mean trapped mode occupation \bar{N} becomes large, we may treat n as a continuous variable and expand the brackets in a power series about \bar{N} . In contributing to the linewidth it will be seen that the correction is of order r/\bar{N}^3 since a further division by \bar{N} occurs in deriving the spectrum. Since we consider the regime $n_s < \bar{N}$ and $1 \ll \bar{N}$ we neglect the correction hereafter. The equation of motion for $\langle a^\dagger \rangle$ can be treated similarly, but in addition it is assumed that as the system is driven further above threshold gain narrowing will occur, permitting the coherent state approximation $\rho \simeq |\sqrt{\bar{N}}\rangle \langle \sqrt{\bar{N}}|$ [11]. The matrix elements become

$$\rho_{n,m} \simeq \frac{e^{-\bar{N}} \bar{N}^{(n+m)/2}}{\sqrt{n!m!}} |n\rangle \langle m| \quad (22)$$

and the equation of motion for $\langle a^\dagger(t) \rangle$ has the pump contribution

$$\begin{aligned} \frac{d}{dt} \langle a^\dagger(t) \rangle \Big|_{\text{pump}} &= r \sum_{n=0}^{\infty} \frac{\sqrt{n} \rho_{n-1,n}}{2(n+n_s)+1} \\ &\simeq \frac{r}{2(\bar{N}+n_s)+1} \langle a^\dagger(t) \rangle \equiv P \langle a^\dagger(t) \rangle. \end{aligned} \quad (23)$$

Expanding the exact expression in a power series about $1/\bar{N}$ and using the coherent state approximation shows the correction is of order r/\bar{N}^3 .

Steady-state occupation in the Born approximation

In order to easily compare our results with the familiar quantum optical case it will be necessary to discuss the Born approximation. The Born approximation assumes that the density matrix ρ can be written as a tensor product of the reduced density matrix for system σ , and the reduced density matrix for the the output field. It further assumes that the output field remains in its original state. If we trace over the output field modes, we obtain a master equation for the lasing mode which has the same pumping term as given above. The reduced density now describes only the cavity mode, and $\sigma_{n,n} = \langle n|\sigma|m\rangle$ are c numbers. The damping term has the form

$$\begin{aligned} (\dot{\sigma})_{\text{damp}} &= -\gamma \int_0^t du \{ f(u) e^{i\omega_o t} [a^\dagger a \sigma(t-u) \\ &\quad - a \sigma(t-u) a^\dagger] + H.c. \}. \end{aligned} \quad (24)$$

Combining the pumping and damping terms we can find the equation of motion for the steady-state trap occupation distribution, $p_n = \sigma_{n,n}$,

$$\begin{aligned} \dot{p}_n(t) &= \frac{rn}{n+n_s} p_{n-1}(t) - \frac{r(n+1)}{n+n_s+1} p_n(t) \\ &\quad - 2n \int_0^t du \operatorname{Re}(f(u) e^{i\omega_o t}) p_n(t-u) \\ &\quad + 2(n+1) \int_0^t du \operatorname{Re}(f(u) e^{i\omega_o t}) p_{n+1}(t-u). \end{aligned} \quad (25)$$

By setting the derivatives to zero and assuming that the functions $p_n(t)$ approach a constant p_n^{ss} in the long time limit, we find the recursion relation

$$\begin{aligned} \frac{rn}{n+n_s} p_{n-1}^{ss} + (n+1) 2\gamma_{BM} p_{n+1}^{ss} \\ - \left(\frac{r(n+1)}{n+n_s+1} - 2n\gamma_{BM} \right) p_n^{ss} = 0, \end{aligned} \quad (26)$$

where

$$\gamma_{BM} = \gamma \int_0^\infty du \operatorname{Re}(f(u) e^{i\omega_o u}), \quad (27)$$

is the effective damping constant. The solution

$$p_n^{ss} = \mathcal{N} \frac{(r/2\gamma_{BM})^n}{(n+n_s)!} \quad (28)$$

is almost identical to that obtained for the optical laser [13]. The distribution is thermal for $r/\gamma_{2BM} < n_s$, and for $r/2\gamma_{BM} \gg n_s$, it approaches a Poissonian distribution with mean \bar{N} and variance V given by

$$\bar{N} = \frac{r}{2\gamma_{BM}} - n_s, \quad (29)$$

$$V = \bar{N} + n_s. \quad (30)$$

Frequency shift and linewidth in the Markov approximation

If the evolution of the system mode is much slower than the timescale of decay for the memory function $f(t)$, the Markov approximation may be used to obtain a useful limit of the equations of motion. Using the solution for $c_p(t)$, we may write

$$\begin{aligned} \frac{da^\dagger(t)}{dt} &= -\frac{i}{\hbar} [a^\dagger, H_p] + i\omega_o a^\dagger(t) + \sqrt{\gamma} \xi^\dagger(t) \\ &\quad - \gamma \int_0^t du a^\dagger(u) f^*(t-u), \end{aligned} \quad (31)$$

where

$$\xi(t) = \int dp \kappa(p)^* c_p(0) e^{-i\omega_p t} \quad (32)$$

is an operator-valued noise term generated by the initial state of the output which serves to preserve the commutation relations. The commutator is given by the memory function found above

$$[\xi(t), \xi^\dagger(t')] = f(t - t'), \quad (33)$$

which decays on a timescale called the memory time [17], which we denote by T_m . This characterizes the timescale for which an output atom may influence the system evolution. The Markov approximation is made by moving to a rotating frame at the system frequency $a(t) = \tilde{a}(t)e^{-i\omega_o t}$, $c_p(t) = \tilde{c}_p(t)e^{-i\omega_o t}$, in which, if T_m is very small compared with the timescale for evolution of the system operator, we may take $\tilde{a}^\dagger(t)$ out of the integrals and obtain the equation of motion

$$\frac{d\tilde{a}^\dagger(t)}{dt} = -\frac{i}{\hbar}[\tilde{a}^\dagger, H_p] + i\Delta_M \tilde{a}^\dagger(t) \quad (34)$$

$$-\gamma_{BM} \tilde{a}^\dagger(t) + \sqrt{\gamma} \tilde{\xi}^\dagger(t), \quad (35)$$

where

$$\Delta_M = \gamma \int_0^\infty du \operatorname{Im}(f(u)e^{i\omega_o u}). \quad (36)$$

We can now construct equations of motion for the occupation and the first order correlation function $g^{(1)}(\tau) \equiv \lim_{t \rightarrow \infty} \langle a^\dagger(t + \tau)a(t) \rangle / \bar{N}$. In the Markov approximation the equations of motion for the occupation and the two time correlation are uncoupled, and the equation of motion for $\langle a^\dagger(t + \tau)a(t) \rangle$ is independent of t . The equations are

$$\frac{d\bar{N}(t)}{dt} = r \left(\frac{\bar{N}(t)}{\bar{N}(t) + n_s} \right) - 2\gamma_{BM} \bar{N}(t), \quad (37)$$

and

$$\frac{d}{dt} g^{(1)}(t) = (P - \gamma_{BM} + i(\omega_o + \Delta_M)) g^{(1)}(t). \quad (38)$$

The steady state two-time correlation is an exponential leading to a Lorentzian spectrum. Using the notation $g^{(1)}(t) = \exp(-\Gamma_M |t| + i(\Delta_M + \omega_o)t)$, the steady state parameters are

$$\bar{N} = \frac{r}{2\gamma_{BM}} - n_s, \quad (39)$$

$$\begin{aligned} \Gamma_M &= \frac{r}{2(\bar{N} + n_s)} - \frac{r}{2(\bar{N} + n_s) + 1} \\ &= \frac{r}{4(\bar{N} + n_s)^2} + O(\bar{N}^{-3}), \end{aligned} \quad (40)$$

and the frequency shift is given by (36). From this point of view it is clear that the Born and Markov approximations are closely related if we are only describing the trap; in particular, the effective damping constants are identical. This is because the output and trap spaces remain uncorrelated in both approximations.

EQUATIONS OF MOTION

We are now in a position to find the non-Markovian equations of motion including the effect of the pump. The equations of motion for the system+output, (15) and (14), are linear and Markovian; however the equations of motion for the system mode alone will depend on the output field dynamics through the memory function $f(t)$. Using the pump terms from Equations (21) and (23), and substituting the formal solution of (14) into (15), the equations of motion for the trapped mode are

$$\frac{d}{dt} \langle a^\dagger a \rangle(t) = r \frac{\bar{N}}{\bar{N} + n_s} - \gamma 2\operatorname{Re} \left(\int_0^t du f(u) \langle a^\dagger(t) a(t-u) \rangle \right), \quad (41)$$

$$\frac{\partial}{\partial \tau} \langle a^\dagger(t + \tau) a(t) \rangle = (i\omega_o + P) \langle a^\dagger(t + \tau) a(t) \rangle - \gamma \int_0^{t+\tau} du f^*(t + \tau - u) \langle a^\dagger(u) a(t) \rangle, \quad (42)$$

and for the output modes we find

$$\frac{d\langle c_p^\dagger c_p \rangle}{dt} = \gamma |\kappa(p)|^2 2\operatorname{Re} \left(\int_0^t du e^{-i\omega_p(t-u)} \langle a^\dagger(t) a(u) \rangle \right). \quad (43)$$

These equations of motion were first derived in [11], with a seemingly minor modification of the pump term in (41). This will be discussed in the following section as it turns out to have important implications for the predictions of the model. The first two coupled integro-differential equations describe the system and output dynamics, starting from when the pump

is initiated. We are primarily interested in the stationary solutions, and once the system reaches a steady state the output energy spectrum will not directly be of interest as it is steadily growing. The long time limit of (43) is the spectrum of the output energy flux which will become stationary. In this case we will see that the relationship between the output flux spec-

trum and the power spectrum of the trapped mode is in keeping with Fermi's golden rule.

The equations of motion (37) and (38) of [11] are identical to our equations (41) and (42) apart from the additional factor $\bar{N}/(\bar{N} + n_s)$ in (41). The treatment of the pump term in [11], where this factor was taken as unity, produced incorrect results because a similar approximation was not made in the pump term in (38) of [11]. We will discuss this inconsistency in more detail once we have derived the stationary equations of motion.

Stationary equations of motion

The stationary solutions of (41) and (42) may be found by breaking up the integrals and using the fact that the memory function vanishes in the long time limit. We will assume the system reaches a steady state at $t = t_{ss}$. The integral in (41) may then be written as

$$\lim_{t \rightarrow \infty} \int_0^t du f(u) \langle a^\dagger(t) a(t-u) \rangle = \lim_{t \rightarrow \infty} \int_0^{t-t_{ss}} du f(u) g^{(1)}(u) \bar{N} + \lim_{t \rightarrow \infty} \int_{t-t_{ss}}^t du f(u) \langle a^\dagger(t) a(t-u) \rangle \quad (44)$$

$$= \int_0^\infty du f(u) g^{(1)}(u) \bar{N}. \quad (45)$$

Similar manipulations for the integral of (42) lead to the term

$$\lim_{t \rightarrow \infty} \int_0^{t+\tau} du f^*(t+\tau-u) \langle a^\dagger(t) a(t-u) \rangle = \int_0^\infty du f(u)^* g^{(1)}(\tau-u) \bar{N}, \quad (46)$$

where we have assumed stationarity when t is much larger than the memory time T_m , defined as the support of the memory function. Using these expression we find the stationary equations of motion for the trapped mode as

$$r = \gamma(\bar{N} + n_s) 2\text{Re} \left(\int_0^\infty du f(u) g^{(1)}(u) \right), \quad (47)$$

$$\frac{d}{d\tau} g^{(1)}(\tau) = (i\omega_o + P) g^{(1)}(\tau) - \gamma \int_0^\infty du f^*(u) g^{(1)}(\tau-u), \quad (48)$$

$$\frac{d\langle c_p^\dagger c_p \rangle}{dt} = 2\pi\gamma\bar{N} |\langle u_p | \kappa \rangle|^2 S(\omega_p). \quad (49)$$

which are exact in the long time limit. Equations (47) and (48) may be solved self consistently for the mean occupation and the two time correlation of the trap mode. The variables are the pumping rate r , the saturation Boson number n_s , the damping rate γ , the memory function $f(t)$, and the system frequency ω_o . One might suspect that the equation for $g^{(1)}(\tau)g^{(1)}(\tau)$ would be amenable to some kind of transform method; however, it is easily seen that this approach does not lead to any major simplifications. If the integral is decomposed into $(\int_0^\tau + \int_\tau^\infty) d\tau$, the first term is a convolution, but the second term is still important for small τ , and this is the region of interest. The present form has the added simplicity that we only require the memory function $f(t)$, and not its Laplace transform which is generally more complicated.

In deriving the output flux expression we have defined the stationary power spectrum of the trapped mode as

$$S(\omega) \equiv \lim_{t \rightarrow \infty} \int_{-t}^t d\tau \frac{\langle a^\dagger(t+\tau) a(t) \rangle}{2\pi \langle a^\dagger a \rangle(t)} e^{-i\omega\tau} \quad (50)$$

$$= \frac{1}{2\pi} \int d\tau g^{(1)}(\tau) e^{-i\omega\tau}. \quad (51)$$

Equation (49) is then readily found from (43) by handling the integrals in a similar manner as for the trapped mode equations. This expression is essentially Fermi's Golden rule for the atom laser model. It is easily verified that this result implies that our stationary results are exact within first order perturbation theory [18]. This is because the coupling only has non-zero matrix elements between the trap and output eigenspaces, so that all higher order perturbation expressions, which arise from energy nonconserving virtual transitions between output eigenstates, vanish identically.

If the spectrum for the trap is an exponential $g^{(1)}(\tau) = \exp(-\bar{\Gamma}|\tau| + i\bar{\omega}\tau)$, corresponding to a Lorentzian spectrum, then in the limit that the system linewidth becomes very narrow the flux spectrum becomes

$$\frac{d\langle c_p^\dagger c_p \rangle}{dt} \rightarrow 2\pi\gamma\bar{N} |\langle u_p | \kappa \rangle|^2 \delta(\omega_p - \bar{\omega}). \quad (52)$$

The amplitude of the output spectral flux is now determined by the matrix elements $|\langle u_p | \kappa \rangle|^2$. Using (49) we may find the corresponding expression for the spatial correlation

$$\begin{aligned} & \frac{d}{dt} \langle \psi^\dagger(x') \psi(x) \rangle \\ &= 2\pi\gamma\bar{N} \int dp \phi_p^*(x') \phi_p(x) |\langle u_p | \kappa \rangle|^2 S(\omega_p), \end{aligned} \quad (53)$$

where $\phi_p(x) \equiv \langle x | u_p \rangle$ is the output eigenfunction in the position representation. In deriving these results the interactions in the output beam and the trap have been neglected; however, because the equation for $\dot{c}_p(t)$ remains linear in $a(t)$ when the interactions for the trapped mode are included, Equations (49) and (53) for the output flux also hold when interactions are significant for the trapped mode. The beam may be considered a true atom laser beam when the spatial coherence length given by (53) is significantly longer than the thermal de-Broglie wavelength of the atoms [7]. It is clear from (53) that this will place restrictions on the spectral width of the trap and the force that acts on the output atoms. As the spectrum approaches a delta function in frequency, the coherence length will be determined by the eigenfunctions that have the same energy as the trapped mode.

Since we are interested in a highly occupied trapped mode our description of the trap should, in the single mode model, include a self-interaction term of the form $H_{\text{coll}} = \hbar C a^\dagger a^\dagger a a$ in the Hamiltonian, where C is the nonlinear interaction strength. In the limit where the interactions dominate, the spectrum becomes

$$g^{(1)}(\tau) = \exp[-\bar{N}(1 - e^{2iC\tau})]. \quad (54)$$

which has the interesting feature of periodic revivals at $\tau = m\pi/C$, for integer m [19]. In this regime the non-Markovian term in (42) will become unimportant. However, there are regimes of interest where the interactions are weak, corresponding to a very weak harmonic confinement, or to relatively low occupation number. In this regime, the Markovian effects will alter the effective occupation, producing significant linewidth and frequency changes in the spectrum.

The Self-Consistent Markov Approximation

Although the traditional Born-Markov approximation cannot be made for atom lasers, as the linewidth becomes narrow there is still a timescale separation between the memory time of the outcoupling process and the temporal coherence of the lasing mode. This allows us to deal with the effects of the system memory using the self-consistent Markov approximation, which was first presented in [11]. This involves assuming the spectrum is Lorentzian and solving for the linewidth and spectral shift self consistently in an appropriate rotating frame. We are now in position to find the sufficient condition for the validity of this approximation in the steady state. Since the form of $g^{(1)}(t)$ in our equations of motion is still unknown, we must

make some assumption about its form, if we are to progress further. However, since we are interested in the regime where linewidth narrowing is expected to occur, we will assume that the memory function $f(t)$ has a short decay time compared with the decay of $g^{(1)}(t)$. In this regime the decay of $g^{(1)}(t)$ may be neglected in the integrals of (47) and (48). This corresponds to the case where the atoms are ejected from the trap rapidly compared with the system coherence time. Making the ansatz $g^{(1)}(t) = \exp(-\Gamma_{SM}|t|) \exp(i(\omega_o + \Delta_{SM})t)$, where Γ_{SM} and Δ_{SM} are real constants, the equations of motion become

$$r = (\bar{N} + n_s) 2\gamma \int_0^\infty du \operatorname{Re} \left(f(u) e^{i(\omega_o + \Delta_{SM})u} \right) \quad (55)$$

$$\begin{aligned} & i\Delta_{SM} - \Gamma_{SM} = P \\ & -\gamma \int_0^\infty du f^*(u) e^{-i(\omega_o + \Delta_{SM})u} e^{\Gamma_{SM}(t-|t-u|)}. \end{aligned} \quad (56)$$

As Γ_{SM}^{-1} becomes smaller than T_m we can ignore the variation of the rightmost factor in the second integral. This approximation explicitly removes any time dependence from the equation, as required for the consistence of our ansatz. If we define the following pair of self-consistent parameters that depend solely on the memory function:

$$\gamma_{SM} = \gamma \int_0^\infty du \operatorname{Re} \left(f(u) e^{i(\omega_o + \Delta_{SM})u} \right) \quad (57)$$

$$\Delta_{SM} = \gamma \int_0^\infty du \operatorname{Im} \left(f(u) e^{i(\omega_o + \Delta_{SM})u} \right), \quad (58)$$

then we can immediately determine the steady-state lasing mode occupation and linewidth of the atom laser:

$$\bar{N} = \frac{r}{2\gamma_{SM}} - n_s \quad (59)$$

$$\begin{aligned} \Gamma_{SM} &= \frac{r}{2(\bar{N} + n_s)} - \frac{r}{2(\bar{N} + n_s) + 1} \\ &= \frac{r}{4(\bar{N} + n_s)^2} + O(\bar{N}^{-3}). \end{aligned} \quad (60)$$

These equations now take a very similar form to the Markov equations, but must be solved self consistently for the system variables in the steady state. The possibility of a frequency shift also arises when the more severe Markov approximation is made, but this only depends on the bare trap frequency, and corresponds to the limit $\Delta_{SM} \ll \omega_o$ of (58). (Note that the linewidth, while taking the same functional form as the usual Born-Markov result, depends on \bar{N} which is determined by (59), and hence will be changed when $\Delta_{SM} \neq \Delta_M$).

Equations (58) and (60) are our main results. Comparison with the Markov expressions shows that the frequency shift for the system mode and the effective damping constant must now be found self consistently. If the output damping is dominated by gravity and the wavefunction of the system state is weakly dependent on \bar{N} the memory function will not depend on the system variables. If the mean field interactions are dominant, the memory function will depend on the system

wavefunction, which depends on \bar{N} . This gives the possibility of novel linewidth and frequency dependence on the pumping.

Level shifts and damping rates

It is worth establishing the connection between our results and the more familiar quantum optical theory.

In terms of the output eigenfunctions, the memory function $f(t)$ can be written as

$$f(t) = \int dp |\langle \kappa | u_p \rangle|^2 e^{-i\omega_p t}. \quad (61)$$

We now evaluate the time integral over $f(t)$, and use

$$\frac{1}{z + i\epsilon} \rightarrow \mathcal{P} \left(\frac{1}{z} \right) - i\pi\delta(z), \quad (62)$$

for $\epsilon \rightarrow 0$, where \mathcal{P} denotes the principal value part of the integral. We find

$$\begin{aligned} & \int_0^\infty du f(u) e^{i(\omega_o + \Delta_{SM})u} \\ &= \pi \int dp |\langle \kappa | u_p \rangle|^2 \delta(\omega_p - \omega_o - \Delta_{SM}) \\ & \quad - i\mathcal{P} \int dp \frac{|\langle \kappa | u_p \rangle|^2}{\omega_p - \omega_o - \Delta_{SM}}. \end{aligned} \quad (63)$$

The damping and frequency shift take the form

$$\Delta_{SM} = \gamma \mathcal{P} \int dp \frac{|\langle \kappa | u_p \rangle|^2}{\omega_o + \Delta_{SM} - \omega_p}, \quad (64)$$

$$\gamma_{SM} = \gamma \pi \int dp |\langle \kappa | u_p \rangle|^2 \delta(\omega_o + \Delta_{SM} - \omega_p). \quad (65)$$

The level shift now takes the familiar form of a principal value integral modified by the fact that it is now an implicit equation that must be solved simultaneously with (65). When $\Delta_{SM} \ll \omega_o$, the shift may be found by neglecting Δ_{SM} in the denominator, and can be more easily evaluated using (58) in the same approximation.

OUTPUT COUPLING

The potentials in the output field may be caused by mean field interactions, gravity, or by a magnetic field. In this section we discuss the memory functions for these process, which describe the influence of the output beam on the trapped mode statistics. While our model is only strictly valid in the noninteracting limit where the trapped mode becomes the ground state of the harmonic potential, we will also find the memory function for the case of a more highly occupied Bose-Einstein condensate, because deriving a simple description of this output coupling process for the Thomas-Fermi regime is of general interest and some practical use.

Gravity

If the Greens function for the output potential is known, and the position space coupling function $\langle x | \kappa, t \rangle$ is simple enough to compute the necessary integrals, the memory function may be found exactly. The memory functions for gravity and the magnetically anti-trapped output state have been found using the Gaussian form for the coupling [17]

$$\kappa(x) = \left(\frac{1}{\pi\sigma^2} \right)^{1/4} e^{-x^2/2\sigma^2}. \quad (66)$$

When the potential is $V(x) = -mgx$ the result is

$$f(t) = \frac{\exp \left\{ -\left(\frac{\omega_g t}{2} \right)^2 - i \left(\frac{mg^2}{24\hbar} \right) t^3 \right\}}{\sqrt{1 + i\omega_k t}}, \quad (67)$$

where the kinetic and potential energies associated with the initial wavepacket are $\hbar\omega_k = \hbar^2/2m\sigma^2$, and $\hbar\omega_g = mg\sigma$ respectively. The general effect of the output coupling is to change the system frequency and the effective damping. To show this for the case of gravitational damping we use $\gamma = 2 \times 10^5 \text{ s}^{-2}$, and find the Markov and self consistent solutions as the pumping is increased. We use $n_s = 47$, and increase the pumping from $2 \times 10^5 \text{ s}^{-1}$ to $2 \times 10^6 \text{ s}^{-1}$. We use an atomic mass of $5 \times 10^{-26} \text{ kg}$. The bare system frequency is taken as $\omega_o = 2\pi \times 123 \text{ Hz}$. We choose gravity as $g = 9.8 \sin(0.1) \text{ ms}^{-2}$. We will also take the width of the coupling to be $\sigma = 1.6 \times 10^{-6} \text{ m}$, which corresponds to the momentum width used in [11]. Figure 2 shows the self consistent steady state occupation and linewidth calculated for these parameters. The Markov approximation neglects the effective weakening of the damping that is caused by a large number of virtual transitions during the output coupling. The more accurate self consistent approach predicts an increase in \bar{N} over the Markov results. The difference in linewidths is a consequence of this effective widening of the system boundary, and arises principally from the difference in \bar{N} .

Comparison with previous work

In Figure 2 we have used parameters corresponding to Table I of [11], which examined linewidth as a function of pumping. The previous treatment used the pump term r instead of $r\bar{N}/(\bar{N} + n_s)$ in (41). When stationary solutions are found, instead of the self consistent Markov result (60), the linewidth becomes

$$\frac{r}{2\bar{N}} - \frac{r}{2(\bar{N} + n_s) + 1} = \frac{r(1 + 2n_s)}{4\bar{N}^2} + O(\bar{N}^{-3}), \quad (68)$$

as can easily be verified using the data in Table I. of [11]. This expression can also be written as

$$\frac{r}{2\bar{N}} - \frac{r}{2(\bar{N} + n_s) + 1} = \Gamma_{SM} (1 + 2n_s) + O(\bar{N}^{-3}), \quad (69)$$

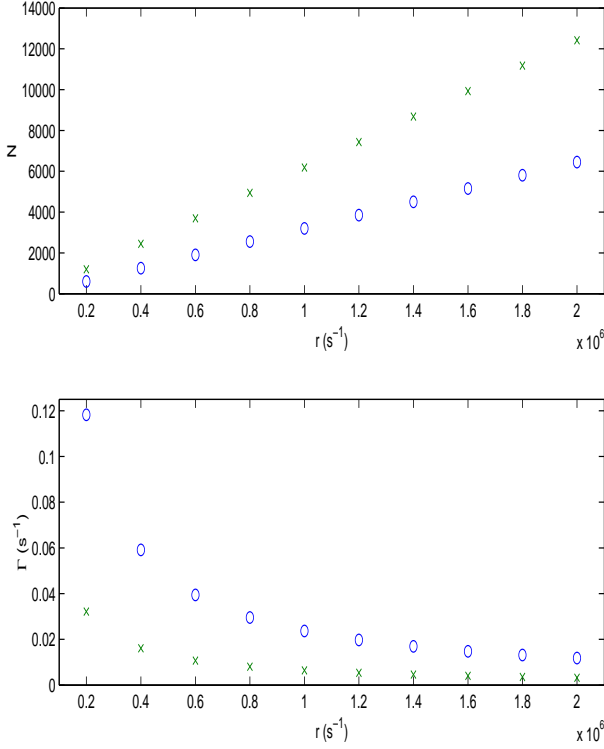


FIG. 2: Numerical results for the Markov approximation (o) and the self consistent approach (x). We use $\gamma = 2 \times 10^5 \text{ s}^{-2}$, $n_s = 47$, $m = 5 \times 10^{-26} \text{ kg}$. The bare system frequency is taken as $\omega_o = 2\pi \times 123 \text{ Hz}$. Gravity is set to $g = 9.8 \sin(0.1) \text{ ms}^{-2}$. The width of the coupling is $\sigma = 1.6 \times 10^{-6} \text{ m}$. a) The steady state occupation number as a function of r . b) The self consistent linewidth as a function of r .

which clearly generates a spurious broadening of the linewidth. This masked the physical behaviour found in the present work, where the linewidth is reduced below the Born-Markov prediction by the non-Markovian evolution.

Mean field interactions and antitrapping

When the output atoms are coupled into an antitrapped state the memory function should be calculated for an inverted parabolic potential. If the atoms also experience significant mean field repulsion from a highly occupied trap mode, the effective output potential will be described by a rescaled inverted parabola corresponding to the Thomas-Fermi solution for the condensate wavefunction. Using the same Gaussian form of the coupling and taking the height of the potential at the center of the trap as $\tilde{V} = \hbar\tilde{\omega}$, the memory function is [17]

$$f(t) = e^{-i\tilde{\omega}t} \left(\cosh \omega_o t + i \left(\frac{\omega_k}{\omega_o} - \frac{\omega_o}{\omega_k} \right) \sinh \omega_o t \right)^{-1/2} \quad (70)$$

When $\omega_o^{-1} \ll t$, this becomes

$$f(t) = e^{-i\tilde{\omega}t - \omega_o t/2} \left(\frac{1}{2} + \frac{i}{2} \left(\frac{\omega_k}{\omega_o} - \frac{\omega_o}{\omega_k} \right) \right)^{-1/2}, \quad (71)$$

from which it is clear that the timescale of decay caused by the kinetic dispersion associated with the curvature of the potential is $2/\omega_o$. As noted in [17], this exponential envelope is evident from the Greens function for the inverted parabolic potential, and occurs regardless of the shape of the interaction region. This is also expected to be the case for the Thomas-Fermi mean field potential because while the exact motion will not be correctly reconstructed by this potential outside the interaction region, it is accurately reproduced where the overlap is calculated.

Finding the output evolution for a particular potential is equivalent to finding the Greens function for the output field Schrodinger equation. This provides rather more information than is needed, since we only require the evolution within the region of overlap with the trap state. To find this, we will write the output field effective Schrodinger equation in terms of amplitude and phase variables. We may then use the Raman-Nath approximation to evaluate the short time evolution when the trap is highly occupied since the phase evolution generated by the mean field will be dominant. We will also find the Raman-Nath time, and the region of validity may then be found by comparing this with the memory time arising from the approximate memory function. We will first write the output wave function as $\Psi(\mathbf{x}, t) \equiv A(\mathbf{x}, t)e^{iS(\mathbf{x}, t)}$, whereby the Schrodinger equation reduces to

$$\frac{\partial A}{\partial t} = \frac{-\hbar}{2m} (2\nabla S \cdot \nabla A + A \nabla^2 S) \quad (72)$$

$$\frac{\partial S}{\partial t} = \frac{\hbar}{2m} \left(\frac{\nabla^2 A}{A} - (\nabla S)^2 \right) - \frac{V(\mathbf{x})}{\hbar}. \quad (73)$$

To obtain the effective one dimensional equation of motion, the radial wavefunction is assumed to be in the ground state of the harmonic trapping potential, with frequency ω_r . In terms of the single particle condensate wavefunction ψ_c , the one dimensional Schrodinger equation for the output atoms is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} \psi + V(z)\psi, \quad (74)$$

where the effective potential is

$$V(z) = \tilde{u}_{12} N |\psi_c|^2 = \epsilon \mu_{TF} \left(1 - \frac{z^2}{L^2} \right) \quad (75)$$

$$\equiv V_c \left(1 - \frac{z^2}{L^2} \right), \quad (76)$$

where this is positive, and zero otherwise. The rescaled interaction strength is $\tilde{u}_{12} = u_{12}/2\pi R^2$, and the width of the ground state is determined by the harmonic oscillator radius $R = \hbar/m\omega_r$. The axial Thomas-Fermi radius is $L =$

$\sqrt{2\mu_{TF}/m\omega_o^2}$ and the ratio of trapped-trapped to trapped-untrapped s-wave scattering lengths is ϵ , which we take as unity in what follows. The effective interaction parameter is rescaled by the transverse cross section, and has dimension length \times energy as required for a one dimensional model. The chemical potential becomes

$$\mu_{TF} = \left(\frac{3\tilde{N}\tilde{u}_{11}}{4} \sqrt{\frac{m\omega_o^2}{2}} \right)^{2/3}. \quad (77)$$

Starting from the density-phase equations for the output evolution and reducing to one dimension, we neglect the curvature term $\nabla^2 A/A$, and take the initial phase as constant in position. The phase equation then depends only on the potential, and may be integrated to derive the memory function

$$f_{TF}(t) = \frac{3}{4} \sqrt{\frac{i\pi}{\omega_c t}} e^{-i\omega_c t} h(\omega_c t), \quad (78)$$

where

$$h(x) = \frac{e^{ix}}{\sqrt{-\pi ix}} + \left(\frac{1+2ix}{2ix} \right) \text{erf}(\sqrt{-ix}), \quad (79)$$

and $\omega_c = V_c/\hbar$. The long time behavior for $1 \ll x$, $h(x) \rightarrow 1$ has been used to factor out the asymptotic form of $f(t)$.

The Raman-Nath time for the validity of this approximation is the timescale of variation of the amplitude A . Using only the potential for the phase evolution, and again using the smoothly varying envelope approximation so that we drop the $\nabla S \cdot \nabla A$ term, we find the short time solution

$$A(z, t) = A(z, 0) \exp \left(-\omega_c \frac{\hbar}{2mL^2} t^2 \right). \quad (80)$$

Defining the kinetic energy associated with the length scale of the Thomas-Fermi profile as $\hbar\omega_k = \hbar^2/2mL^2$, the Raman-Nath time is

$$\tau_{RN} = \frac{1}{\sqrt{\omega_c \omega_k}} = \frac{2}{\omega_o}. \quad (81)$$

This is just the decay caused by the kinetic energy arising from the curvature of the potential, as seen in our previous discussion for the anti-trapped Gaussian wavepacket. Note that this does not depend on \tilde{N} because the Thomas-Fermi solution always assumes the form of an inverted image of the trap with amplitude given by the chemical potential. We can find the condition for the validity of the approximate memory function for our purposes by comparing this timescale with the memory time. The memory time may be defined by the ratio [17]

$$R = \frac{|\int_{T_m}^{\infty} dt f(t)|}{|\int_0^{\infty} dt f(t)|}. \quad (82)$$

When T_m is chosen so that the ratio R is small, the contribution of any time integral equation involving $f(t)$ will be significant for times less than T_m . We want to find T_m when $R \ll 1$ for the Thomas-Fermi memory function (78). We have

$$R = \frac{\sqrt{\pi}}{2} \left| \frac{e^{-i\omega_c T_m} \text{erf}(\sqrt{-i\omega_c T_m})}{\sqrt{-i\omega_c T_m}} \right|. \quad (83)$$

We now assume $1 \ll \omega_c T_m$, and use the asymptotic form of the memory function [20] to find

$$\frac{\pi}{4\omega_c R^2} \lesssim T_m. \quad (84)$$

Using $R = 10^{-2}$, the condition $T_m \ll \tau_{RN}$ becomes

$$\omega_o \ll 10^{-3} \omega_c. \quad (85)$$

When this condition holds the approximate memory function will determine the effect of the output coupling on the system evolution.

Kinetic evolution

We are primarily interested in the regime where the phase evolution determines the memory function. However, if we include the long-time exponential decay determined by the kinetic evolution, the analysis is greatly simplified. This is because neglecting the kinetic evolution entirely leads to an unphysical long-time behavior of the memory function, which renders the equations of motion insoluble. This feature is related to the existence of a bound state in the dressed states of the trap plus output potentials, which has been discussed in detail elsewhere [11]. Since there is a separation of timescales, we may include the kinetic evolution by simply multiplying the approximate memory function by $e^{-\omega_o t/2}$. This has no effect on the overlap $f(t)$ over the region in which the integrand is significant, but restores the description of the long time behavior to a more realistic form. The memory function that we will use takes the final form

$$f_{TF}(t) = \frac{3}{4} \sqrt{\frac{i\pi}{\omega_c t}} h(\omega_c t) e^{-i\omega_c t - \omega_o t/2}. \quad (86)$$

Mean field damping in the Markov approximation

We will now find the linewidth and the frequency shift of the atom laser model using $f_{TF}(t)$ in the Markov approximation, and show that mean field damping leads to an interesting scaling behavior for the trap. We can compute an analytic solution by using the integral

$$\int_0^\infty dt f_{TF}(t) e^{-st} = \frac{3i}{2\omega_c} \left(\frac{i(s + \omega_o/2)}{\sqrt{\omega_c(i s + i\omega_o/2 - \omega_c)}} \arctan \sqrt{\frac{\omega_c}{i s + i\omega_o/2 - \omega_c}} - 1 \right). \quad (87)$$

This is easily found from the integral representation

$$f_{TF}(t) = \frac{3}{2} \int_0^1 dx (1 - x^2) e^{-i\omega_c t(1-x^2)} e^{-\omega_o t/2}, \quad (88)$$

which compactly expresses the approximations we are using for the output evolution.

We now use the system frequency $\omega_o = \omega_c$ for the condensate, and find that when $\omega_o \ll \omega_c$ the asymptotic form of \arctan [20] leads to

$$\Delta_M = \gamma_M = \frac{3\pi\gamma}{4\sqrt{\omega_c\omega_o}}. \quad (89)$$

The occupation becomes

$$\bar{N} = \frac{2r\sqrt{\omega_c\omega_o}}{3\pi\gamma} - n_s, \quad (90)$$

so that with the \bar{N} dependence of $\omega_c = \mu_{TF}/\hbar$, well above threshold the trap number becomes

$$\bar{N} = C \left(\frac{r}{\gamma} \right)^{3/2}, \quad (91)$$

where

$$C = \left(\frac{2}{3\pi} \sqrt{\frac{\omega_o}{\hbar}} \right)^{3/2} \left(\frac{3\tilde{u}_{11}}{4} \sqrt{\frac{m\omega_o^2}{2}} \right)^{1/2}. \quad (92)$$

This is a scaling behaviour quite different to the optical case which varies as $\bar{N} \sim r/\gamma$. The trap spectrum linewidth is

$$\Gamma_M = \frac{r}{4\bar{N}^2} = \frac{\gamma^3}{4(rC)^2}. \quad (93)$$

The possibility of the the damping depending on \bar{N} leads to some interesting new scaling behavior. The expression (91) demonstrates that the Thomas-Fermi profile is not a particularly good repelling potential, requiring that if the flux is increased, the occupation must adjust by a correspondingly larger amount in order to repel the extra atoms rapidly enough to recover equilibrium. The linewidth scales as γ^3/r^2 , rather than the usual optical scaling of γ^2/r which has consequences for the response of atom laser statistics to fluctuations.

The validity of the Markov approximation is determined by the condition $\Delta \ll \omega_c$, which leads to the requirement

$$\left(\sqrt{\frac{2}{m\omega_o^3}} \frac{\pi\gamma}{\tilde{u}_{11}} \right)^{2/3} \ll \bar{N}. \quad (94)$$

This forces a restriction on the rate of output coupling relative to the interaction strength and the occupation.

CONCLUSIONS

We have carried out a non-Markovian steady state analysis of a fully quantum mechanical atom laser model. We have demonstrated that a self-consistent Markov approximation is valid provided the laser is operating in a linewidth narrowing regime and the reservoir correlation time is sufficiently short. We have shown that the difference between the Born-Markov approximation and the more exact treatment is that the system frequency, occupation number and effective damping may be shifted by the coupling process. This leads to corrections in the effective damping rate and the steady state trap occupation number. We have found a simple analytical form for the memory function for a Bose-Einstein condensate output coupler in the Thomas-Fermi regime, which gives rise to nonlinear scaling of the steady state occupation with pumping rate. This has implications for the design of linewidth narrowing quantum feedback schemes, and for the stability of atom lasers.

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